COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **22** (2) (2002) 217–244

SPARSE QUASI-RANDOM GRAPHS

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Received March 22, 1999

1. Introduction

Quasi-random graph properties form a large equivalence class of graph properties which are all shared by random graphs. In recent years, various aspects of these properties have been treated by a number of authors (e.g., see [5]-[14], [16], [23]-[27]). Almost all of these results deal with *dense* graphs, that is, graphs on n vertices having cn^2 edges for some $c>0$ as $n\to\infty$. In this paper, we extend our study of quasi-randomness to sparse graphs, i.e., graphs on n vertices with $o(n^2)$ edges. It will be shown that many of the quasi-random properties for dense graphs have analogues for sparse graphs, while others do not, at least not without additional hypotheses. In general, sparse graphs are more difficult to deal with than dense graphs, due for example to the possible absence of certain local structures, such as 4-cycles.

2. Notation and background

If G is a graph with vertex set $V(G)$ and edge set $E(G)$, we will let $v(G):=|V(G)|$ and $e(G):=|E(G)|$. Our graphs will (usually) be undirected with no loops and no multiple edges. The notation $G(n)$ will denote a graph G with n vertices. If $\{x,y\}\in E(G)$ is an edge of G, we will write $x\sim y$. The *neighborhood* $Nd(x)$ of x is given by

$$Nd(x) := \{y \in V(G) : x \sim y\}.$$

Mathematics Subject Classification (2000): 05C80

[1] Research supported in part by NSF Grant No. DMS 98-01446

0209–9683/102/$6.00

The *degree* $\deg x$ of a vertex x is just $|Nd(x)|$. If $X \subset V(G)$, we denote by $e(X)$ the number of edges $x \sim y$ in G with $x, y \in X$, i.e., the number of edges in the subgraph $G[X]$ of G induced by X. The *adjacency matrix* $A = A(G)$ of G is the square matrix indexed by the vertices of G with $A(x,y) = \{\begin{smallmatrix} 1 \text{ if } x \sim y, \\ 0 \text{ otherwise.}\end{smallmatrix}$ Since A is real and symmetric, A has real eigenvalues, which we denote by $\lambda_i = \lambda_i(G)$ where we assume $v(G) = n$ and

$$\lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

By a well-known theorem of Frobenius, λ_1 is positive with an eigenvector with non-negative coordinates.

For graphs G and H, we denote by $\#\{H \subset G\}$ the number of occurrences of H as a (labelled) *subgraph* of G. That is, $\#\{H \subset G\} := |\{\rho : V(H) \to V(G)$ such that $x \sim y$ in $H \Rightarrow \rho(x) \sim \rho(y)$ in $G\}|$. Note that ρ is not required to be injective. Similarly, $\#\{H < G\} := |\{\rho : V(H) \to V(G)$ such that $x \sim y$ in H if and only if $\rho(x) \sim \rho(y)$ in $G\}|$ will denote the number of occurrences of H as an *induced subgraph* of G. For a function $p = p(n) \in [0,1]$, we let $G_p(n)$ denote a *random graph* with edge probability p. This actually denotes a distribution on the set of graphs on n vertices, which we will usually take to be the set $[n] := \{1, 2, \ldots, n\}$, in which each pair $\{i, j\}$ is selected to be an edge independently with probability p. The most commonly studied graphs are $G_{1/2}(n)$.

Finally, we will let $\mathcal{G}_p$ denote an infinite family of graphs $\{G(n) : n \to \infty\}$ with the property that $e(G(n)) = (1 + o(1))p\binom{n}{2}$ as $n \to \infty$.

The basic theme of quasi-randomness is illustrated by the following description. Consider the following five properties that the graphs $G = G(n)$ in a family $\mathcal{G}_{1/2}$ might have:

(I) For all $X \subset V(G)$,

$$e(X) = \frac{1}{4}|X|^2 + o(n^2);$$

(II) $\#\{C_4 \subset G\} \leq (1 + o(1))\frac{n^4}{16}$, where C_4 denotes a 4-cycle;

(III)

$$\sum_{u,v} \left| |Nd(u) \cap Nd(v)| - \frac{n}{4} \right| = o(n^3).$$

(IV) For any fixed graph $H = H(t)$,

$$\#\{H < G\} = (1 + o(1))n^t 2^{-\binom{t}{2}};$$

(V)

$$\lambda_1(G) = (1 + o(1))n/2, \quad \lambda_i(G) = o(n) \text{ for } i \geq 2.$$

It is well-known that almost all random graphs $G_{1/2}(n)$ possess properties (I–V). What is perhaps less obvious is that these properties are in fact *equivalent*, in the sense that *any* family $\mathcal{G}_{1/2}$ of graphs which satisfies *any one* of the properties must in fact satisfy *all* of them. Nevertheless, these and a variety of other properties have now been shown to be equivalent (e.g., see [10,12]). We call families of graphs which satisfy this equivalence class of properties *quasi-random*. In particular, as a consequence, a number of rather simple ways are now available for explicitly constructing graphs which imitate random graphs from this point of view.

In this note, we focus on extending these ideas to graphs which are much sparser, e.g., having $e(G) = o(n^2)$ edges. We point out that there are some results already available in the literature which have this flavor, most notably, the seminal results of Thomason [26–28] on (p,α)-jumbled graphs. In these graphs, however, rather more stringent hypotheses are required on the discrepancy of the behavior of parameters under investigation in order to reach the desired conclusions. In particular, the requirement that a graph G be (p,α)-jumbled is that for all $X \subset V(G)$,

$$\left| e(X) - p\binom{|X|}{2} \right| \leq \alpha|X|$$

with α (usually) required to satisfy $\alpha = o(p^2 n)$ [28, p. 178]. Thus, for example, if $|X| = n/2$ and $p = n^{-1/2}$, the "error term" $\alpha|X|$ is required to be of the order $o(p^2 n^2) = o(n)$, much smaller than the number of edges $p\binom{|X|}{2} \sim \frac{1}{8}n^{3/2}$. The point is that for this definition the error from the expected behavior is required to be of a strictly smaller growth rate, e.g., n versus $n^{3/2}$. Of course, random graphs typically have this behavior, and when it holds, correspondingly stronger conclusions follow (e.g., see [26–28]). Some nice recent examples of this behavior also appear in Alon and Krivelevich [1].

Our approach will only require that the actual value μ of a particular parameter should differ from what one would expect in a random graph by a term of the form $o(\mu)$ (as opposed to smaller terms such as $o(\mu^{1/2})$, $o(\mu/\log\mu)$, etc.). Naturally, not as much typically follows from these relaxed assumptions. However, by doing this, we are able to identify a rich class of equivalent properties which have numerous applications to situations where only these somewhat less stringent properties of truly random graphs are needed.

3. Basic properties

We first list a number of properties for any family $\mathcal{G}_p$ of graphs, which are straightforward generalizations of the properties (I)–(IV) in the preceding section. It will turn out that for sufficiently small p, (e.g., $p = p(n) = o(n^{-1/2})$), these properties are not all equivalent. Nevertheless, we will show that under strengthened hypotheses, we can still identify an interesting analogous class of *t-quasi-random* properties. The description of these properties all involve $o(\cdot)$ terms. For two properties $P = P(o(1))$ and $P' = P'(o(1))$, the statement $P(o(1)) \Rightarrow P'(o(1))$ means that for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that $P(\delta) \Rightarrow P'(\epsilon)$. Two properties are said to be equivalent if $P \Rightarrow P'$ and $P' \Rightarrow P$.

In what follows we assume $p = p(n)$ with $pn \to \infty$ as $n \to \infty$, and $\mathcal{G}_p = \{G(n) : n \to \infty\}$ is a family of graphs with $e(G(n)) = (1 + o(1))p\binom{n}{2}$. For notational simplicity we will often write $G = G(n)$ and $V = V(G)$ (so that $|V| = n$). Here we define the following properties that we will consider for $\mathcal{G}_p$:

DISC: For all $X \subset V$,

$$\left| e(X) - p\binom{|X|}{2} \right| = o(pn^2).$$

(DISC stands for "discrepancy".) A related (and, we will show, equivalent) condition is:

DISC(1)**:** For all $X, Y \subset V$,

$$|e(X, Y) - p|X|\,|Y|| = o(pn^2).$$

For $X, Y \subset V$, let $e_t(X, Y)$ denote the number of walks $v_0, v_1, \ldots, v_t$ with v_0 in X and v_t in Y, and with $v_i \sim v_{i+1}$ for $0 \le i < t$. Note that here we do not require that all vertices or edges in a walk to be distinct. In contrast, we define a *path* to be a walk in which all vertices are distinct. Similarly, let $P_t(X, Y)$ denote the set of paths $v_0, v_1, \ldots, v_t$ with $v_0 \in X$ and $v_t \in Y$. To simplify the notation, we write $e(X, Y) = e_1(X, Y)$, and $e_t(x, y) = e_t(\{x\}, \{y\})$. Throughout this paper t denotes a fixed positive integer.

DISC(t)**:** For all $X, Y \subset V$,

$$|e_t(X, Y) - p^t n^{t-1}|X|\,|Y|| = o(p^t n^{t+1}).$$

EIG: The eigenvalues $\lambda_i = \lambda_i(G)$, $\lambda_1 \ge |\lambda_2| \ge \cdots \ge |\lambda_n|$, satisfy

$$\lambda_1 = (1 + o(1))pn,$$
$$\lambda_i = o(pn), \; i > 1.$$

A related condition is the following (where t is a positive integer):

EIG (t): The eigenvalues $\lambda_i = \lambda_i(G)$ of G satisfy

$$\sum_i |\lambda_i|^t = (1+o(1))(pn)^t.$$

By a *t-circuit* C_t^* we mean a walk $x_0, x_1, \dots, x_t$ of length t with $x_0 = x_t$.

CIRCUIT (t): The number of t-circuits C_t^* in G satisfies

$$\#\{C_t^* \subset G\} = (1+o(1))(pn)^t.$$

A *cycle* is a circuit in which all the vertices are required to be distinct, so that a t-cycle has t distinct vertices. An analogous condition for t-cycles is the following:

CYCLE (t): The number of t-cycles C_t in G satisfies

$$\#_{1-1}\{C_t \subset G\} = (1+o(1))(pn)^t$$

where we define $\#_{1-1}\{C_t \subset G\} = \{\rho : V(C_t) \to V(G)$ such that $x \sim y$ in C_t implies $\rho(x) \sim \rho(y)$ in G and ρ is injective $\}$.

We point out here that the common value for EIG $(2t)$, CIRCUIT $(2t)$ and CYCLE $(2t)$, namely $(1+o(1))(pn)^{2t}$, is what you expect in a p-random graph $G_p(n)$ when $p \gg n^{-1+1/t}$ (i.e., when $pn^{1-1/t} \to \infty$ as $n \to \infty$). When p is much smaller, however, then this is not what happens in a p-random graph. For example, when $p = n^{-3/4}$, then $G_p(n)$ would be expected to have $(1+o(1))p^2n^3 = (1+o(1))n^{3/2}$ 4-circuits, which is a lot more than $(1+o(1))(pn)^4 = (1+o(1))n$, the expected number of 4-cycles in $G_p(n)$.

Finally, we have the (very weak) condition:

AR:

$$\sum_{x \in V} |\deg x - pn| = o(pn^2)$$

(i.e., G is "almost regular".)

Theorem 1. *For any family $\mathcal{G}_p$ with $pn \to \infty$ as $n \to \infty$, the following implications hold for all $t \geq 1$:*

$$\begin{array}{ccccccc} \text{CIRCUIT}\,(2t) \Leftrightarrow \text{EIG}\,(2t) \Rightarrow & \text{EIG} & \Rightarrow \text{DISC} \Leftrightarrow \text{DISC}\,(1) \\ & \Downarrow & \Downarrow \\ & \text{DISC}\,(t) & \text{AR} \end{array}$$

The various implications in Theorem 1 will be useful later for establishing quasi-random classes of properties for $\mathcal{G}_p$.

4. Some useful facts

The proof of Theorem 1 consists of a series of basic facts:

Fact 1: EIG $\Rightarrow$ AR.

Proof. Let $\bar{v} := (1,1,\ldots,1)^*$ (where $*$ denotes transpose). For the adjacency matrix A of G with eigenvalues $\lambda_1 \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$, we have

$$\|A\bar{v}\| \leq \lambda_1 \|\bar{v}\|.$$

Thus,

$$\sum_x (\deg x)^2 \leq (1+o(1))p^2 n^3.$$

On the other hand,

$$e(G) = \frac{1}{2}\sum_x \deg x \geq (1+o(1))p\binom{n}{2}.$$

by the hypothesis on $\mathcal{G}_p$. By Cauchy–Schwarz we have

$$(1+o(1))p^2n^3 \geq \sum_x (\deg x)^2 \geq \frac{1}{n}\left(\sum_x \deg x\right)^2 = (1+o(1))p^2n^3.$$

which implies

(1) $$\sum_x (\deg x - pn)^2 = o(p^2n^3).$$

Applying Cauchy-Schwarz again, we obtain

$$\sum_x |\deg x - pn| = o(pn^2)$$

as required. ∎

Let us denote by $\bar{e}_1$ the eigenvector associated with λ_1, normalized so that $\|\bar{e}_1\| = 1$. The following observation will be useful in what follows.

Fact 2: Assume that EIG holds. Then

$$\|\bar{u} - \bar{e}_1\| = o(1) \quad \text{where} \quad \bar{u} = \frac{1}{\sqrt{n}}(1,1,\ldots,1)^*.$$

Proof. Suppose we write $\bar{u} = \sum_i a_i \bar{e}_i$ where the $\bar{e}_i$ are orthonormal eigenvectors of A. Thus,

$$A\bar{u} = \sum_i a_i \lambda_i \bar{e}_i.$$

By Fact 1 and (1), we have $A\bar{u} = pn\bar{u} + \bar{w}$ where $\|\bar{w}\| = o(pn)$. Consequently,

$$\bar{w} = \sum_i a_i \lambda_i \bar{e}_i - pn\bar{u} = \sum_i (\lambda_i - pn) a_i \bar{e}_i$$

and so

$$\sum_i (\lambda_i - pn)^2 a_i^2 = o(p^2 n^2).$$

Hence, by the assumption that EIG holds,

$$\sum_{i \neq 1} a_i^2 = o(1).$$

Since $\bar{u} = a_1 \bar{e}_1 + \bar{w}_1$ with $\|\bar{w}_1\| = o(1)$ and $\|\bar{u}\| = \|\bar{e}_1\| = 1$, then $|a_1| = 1 + o(1)$. By choosing the sign appropriately, this implies

$$a_1 = 1 + o(1)$$

and Fact 2 is proved. ∎

Fact 3: EIG ⇒ DISC

Proof. For $S \subset V$, define $f_S(x) := \{ \begin{smallmatrix} 1 \text{ if } x \in S, \\ 0 \text{ otherwise.} \end{smallmatrix}$

Then

$$2e(S) = \langle f_S, A f_S \rangle.$$

Suppose we write $f_S = \sum_i b_i \bar{e}_i$. Then

$$\begin{aligned} b_1 &= \langle f_S, \bar{e}_1 \rangle = \langle f_S, \bar{u} + \bar{w} \rangle \\ &= \frac{|S|}{\sqrt{n}} + \langle f_S, \bar{w} \rangle \\ &= \frac{|S|}{\sqrt{n}} + O(\|f_S\| \ \|\bar{w}\|) \\ &= \frac{|S|}{\sqrt{n}} + o(\sqrt{|S|}) \end{aligned}$$

by using Fact 2. Therefore

$$|\langle f_S, A f_S \rangle - \lambda_1 b_1^2| \leq |\lambda_2| \sum_i b_i^2 = |\lambda_2| \ \|f_S\|^2.$$

and

$$\left| 2e(S) - pn\frac{|S|^2}{n} + o(pn|S|) \right| \leq |\lambda_2| \ |S|,$$

i.e.,

$$\left| 2e(S) - p|S|^2 \right| = o(pn^2),$$

which is just DISC. ∎

Fact 4: EIG $\Rightarrow$ DISC(t) for any positive integer t.

Proof. We first note that

$$e_t(X,Y) = \langle f_X, A^t f_Y\rangle.$$

Let us write $f_X = \sum_j a_j \bar{e}_j$ and $f_Y = \sum_i b_i \bar{e}_i$. Then,

$$\begin{aligned}|\langle f_X, A^t f_Y\rangle - \lambda_1^t a_1 b_1| &\le |\lambda_2|^t \sum_{j=2}^{n} |a_j b_j| \\ &\le |\lambda_2|^t \|f_X\|\ \|f_Y\|.\end{aligned}$$

Hence,

$$\left| e_t(X,Y) - p^t n^t \frac{|X|\ |Y|}{n} + o(p^t n^{t+1})\right| \le |\lambda_2|^t \sqrt{|X|\ |Y|}$$

and so,

$$\left| e_t(X,Y) - p^t n^t \frac{|X|\ |Y|}{n}\right| \le o(p^t n^{t+1})$$

which is just DISC(t). ∎

Fact 5: DISC $\Leftrightarrow$ DISC(1)

Proof. First observe that DISC(1) $\Rightarrow$ DISC since DISC is a special case of DISC(1). To see that DISC $\Rightarrow$ DISC(1), take $X, Y \subset V$ and set $x = |X|$, $y = |Y|$. We first suppose $X \cap Y = \emptyset$. Then

$$\begin{aligned}2e(X,Y) &= e(X\cup Y, X\cup Y) - e(X,X) - e(Y,Y) \\ &= ((x+y)^2 - x^2 - y^2)p + o(pn^2) \\ &= 2xyp + o(pn^2)\end{aligned}$$

as required.

Now, suppose $X \cap Y \neq \emptyset$. Write $a = |X \setminus Y|$, $b = |Y \setminus X|$ and $c = |X \cap Y|$. Then

$$\begin{aligned}e(X,Y) &= e(X\setminus Y, Y\setminus X) \\ &\quad + e(X\setminus Y, X\cap Y) + e(X\cap Y, Y\setminus X) + e(X\cap Y, X\cap Y) \\ &= (ab + ac + cb + c^2)p + o(pn^2) \\ &= (a+c)(b+c)p + o(pn^2) \\ &= |X|\ |Y| p + o(pn^2)\end{aligned}$$

as required. ∎

Fact 6: DISC (1) $\Rightarrow$ AR

Proof. Define $V_+ = \{x : \deg x \geq pn\}$ and $V_- = \{x : \deg x < pn\}$. Then,

$$\begin{aligned}\sum_x |\deg x - pn| &= \sum_{x\in V_+} |\deg x - pn| + \sum_{x \in V_-} |\deg x - pn| \\ &= \sum_{x\in V_+} (\deg x - pn) + \sum_{x\in V_-} (pn - \deg x) \\ &= e(V_+, V) - pn|V_+| + pn|V_-| - e(V_-, V) \\ &= o(pn^2)\end{aligned}$$

by the assumption that DISC(1) holds. ∎

Fact 7: CIRCUIT (2t) $\Leftrightarrow$ EIG (2t)

Proof. By the hypothesis of CIRCUIT (2t),

$$\#\{C_{2t}^* \subset G\} = (1 + o(1))(pn)^{2t}.$$

Consider the trace $Tr\ A^{2t}$ of the matrix A^{2t}. The (k,k)-entry $A^{2t}(k,k)$ of A^{2t} satisfies

$$A^{2t}(k,k) = \#\{C_{2t}^* \subset G : C_{2t}^* \text{ starts at vertex } k \text{ of } G\}$$

so that

$$TrA^{2t} = \sum_k A^{2t}(k,k) = \#\{C_{2t}^* \subset G\}.$$

Therefore

$$\#\{C_{2t}^* \subset G\} = (1 + o(1))(pn)^{2t}$$

if and only if

$$TrA^{2t} = \sum_{i=1}^{n} \lambda_i^{2t} = (1 + o(1))(pn)^{2t},$$

i.e.,

CIRCUIT (2t) $\Leftrightarrow$ EIG (2t). ∎

Fact 8: For any positive integer t, EIG (2t) $\Rightarrow$ EIG

Proof. Since

$$\lambda_1^{2t} \le \sum_i \lambda_i^{2t} = (1+o(1))(pn)^{2t},$$

we have

$$\lambda_1 \le (1+o(1))pn.$$

On the other hand, for $\bar{u} = \frac{1}{\sqrt{n}}(1,1,\ldots,1)^*$, we have

$$\begin{aligned}\lambda_1 = \sup_x \frac{\langle x, Ax\rangle}{\langle x, x\rangle} &\ge \langle \bar{u}, A\bar{u}\rangle \\ &= \frac{2}{n}e(G) \\ &= (1+o(1))pn\end{aligned}$$

by the assumption on $e(G)$, and a standard characterization of λ_1. Therefore,

$$\lambda_1 = (1+o(1))pn.$$

Since EIG$(2t)$ implies

$$\sum_i \lambda_i^{2t} = (1+o(1))(pn)^{2t}$$

then we see that

$$\lambda_i = o(pn), \quad i \ge 2.$$

This proves that EIG holds. ∎

Combining Facts 1–8, the proof of Theorem 1 is complete.
We note for later use the following:

Fact 9: For any positive integer t, EIG$(2t)$ $\Rightarrow$ EIG$(2t+2)$

Proof. In the proof of Fact 8, we showed that EIG$(2t)$ implies that $\lambda_1 = (1+o(1))pn$. Hence, if

$$\sum_i \lambda_i^{2t} = (1+o(1))(pn)^{2t}$$

then

$$\begin{aligned}\sum_i \lambda_i^{2t+2} &\le \lambda_1^2 \sum_i \lambda_i^{2t} \\ &= (1+o(1))(pn)^2(pn)^{2t} \\ &= (1+o(1))(pn)^{2t+2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_i \lambda_i^{2t+2} &\geq \lambda_1^{2t+2} \\ &= (1+o(1))(pn)^{2t+2}.\end{aligned}$$

Thus, EIG $(2t+2)$ holds. ∎

5. Separation of properties

In this section we present examples of graph families which satisfy certain of the preceding properties but not others. In particular, these examples are particularly useful for examining the one-way implications in Theorem 1. We first consider the following constructions:

Let $G=(V,E)$ denote a regular connected graph with eigenvalues $\lambda_1 \geq |\lambda_2| \geq \dots$. For a given positive integer m, let $G^{[m]}$ denote the graph which is formed by replacing each vertex v of G by m copies of v, denoted by v_i, $i=1,\dots,m$, where we assume that u_i is adjacent to v_j if and only if u and v are adjacent in G. (This construction was first suggested to us by Tim Gowers [17].) Suppose $G^{[m]}$ has eigenvalues $\tilde{\lambda}_1 \geq |\tilde{\lambda}_2| \geq \dots$. We will show the following:

Fact 10: Let G and $G^{[m]}$ be defined as above with eigenvalues λ_i and $\tilde{\lambda}_i$, respectively. Then,
(i) $\tilde{\lambda}_1 = m\lambda_1$, $\sup_{i\neq 1} |\tilde{\lambda}_i| = m \sup_{i\neq 1} |\lambda_i|$,
(ii) $m\lambda_i$ is an eigenvalue of $G^{[m]}$ for all i.

Proof. Let $\tilde{A}$ denote the adjacency matrix of $G^{[m]}$. Obviously, $G^{[m]}$ is regular so the all 1's vector $\bar{1}$ is an eigenvector of $\tilde{A}$ with eigenvalue $\tilde{\lambda}_1 = m\lambda_1$.

Let φ denote an eigenvector of $\tilde{A}$ associated with the eigenvalue $\tilde{\lambda}$ which is not $\tilde{\lambda}_1$. From the Perron–Frobenius theorem, there is a unique eigenvector (up to a scalar multiple) for $\tilde{A}$ with all coordinates nonnegative, which we can take to be $\bar{1}$. Thus, φ is orthogonal to $\bar{1}$.

Let x_1 and x_2 denote two copies of the vertex x in G. We have

$$\begin{aligned}\tilde{A}\varphi(x_1) &= \sum_{y\sim x} \varphi(y) \\ &= \tilde{A}\varphi(x_2)\end{aligned}$$

so that

$$\tilde{\lambda}\varphi(x_1) = \tilde{\lambda}\varphi(x_2).$$

If $\tilde{\lambda}\neq 0$, we have $\varphi(x_1)=\varphi(x_2)$. Defining $\varphi_0(x)=\varphi(x_i)$, we have

$$\begin{aligned}\tilde{\lambda}^2 &= \frac{\|\tilde{A}\varphi\|^2}{\|\varphi\|^2}\\ &= \frac{\sum_{x,i}\left(\sum_{y\sim x}\sum_j \varphi(y_j)\right)^2}{\sum_{x,i}\varphi(x_i)^2}\\ &= \frac{m^3\sum_x\left(\sum_{y\sim x}\varphi(y)\right)^2}{m\sum_x \varphi(x)^2}\\ &= m^2\frac{\|A\varphi_0\|^2}{\|\varphi_0\|^2}\\ &\leq m^2\sup_{g\perp 1}\frac{\|Ag\|^2}{\|g\|^2}\\ &= m^2|\lambda_2|^2.\end{aligned}$$

Hence,

$$|\tilde{\lambda_2}| \leq m|\lambda_2| = m\sup_{i\neq 1}|\lambda_i|.$$

In the other direction, we see that for any eigenvector ϕ of G associated with the eigenvalue λ, we can extend ϕ to $\phi' : V(G^{(m)}) \to \mathbb{R}$ by taking $\phi'(x_i)=\phi(x)$. Clearly, ϕ' has eigenvalue $m\lambda$. Thus we have

$$|\tilde{\lambda_2}| = \sup_{i\neq 1}|\tilde{\lambda}_i| \geq m|\lambda_2|.$$

This shows that

$$|\tilde{\lambda_2}| = m|\lambda_2|$$

as desired. ∎

Example 1: EIG $\not\Rightarrow$ EIG (4)

Proof. We will first construct a graph H with about $n^{1/5}$ vertices and $n^{3/10}$ edges. In addition, H will have dominant eigenvalue $\rho_1\sim n^{1/10}$ and all other eigenvalues of size at most $c'n^{1/20}$. (Here, c, c', c'', ... denote appropriate constants). In fact, H will have at least $c''n^{1/5}$ eigenvalues of size greater than $c'''n^{1/20}$. (This can be done, for example, by using coset graphs described later in Section 8).

Now, for such a graph H, we consider the graph $G = H^{(m)}$ where we take $m \sim n^{4/5}$. The eigenvalues λ_i of G satisfy EIG since $|\lambda_2| = m|\rho_2| \leq$

$c_1 n^{4/5} n^{1/20} = o(m\rho_1)$. However, EIG (4) is not satisfied since

$$\begin{aligned}\sum_{i>1} \lambda_i^4 &\geq cn^{1/5}(mn^{1/20})^4 \\ &= cn^{18/5} \\ &\neq o(\lambda_1^4).\end{aligned}$$

∎

Example 2: AR $\not\Rightarrow$ EIG, and AR $\not\Rightarrow$ DISC.

Proof. Any regular graph satisfies AR, but it does not have to satisfy EIG or DISC. ∎

Example 3: DISC (1) $\not\Rightarrow$ DISC (2)**.**

Proof. We consider a graph G consisting of a random graph on $n-1$ vertices with edge density $p = n^{-2/3}$ together with one vertex w which is adjacent to all other vertices. Clearly, G satisfies DISC (1). However, it can be easily checked that because of w, we have $e_2(X,Y) \geq |X|\,|Y| \neq o(p^2n^3)$ when $|X|$ and $|Y|$ are both greater than cn, for example. Thus, G does not satisfy DISC (2). ∎

In the other direction, we do not know if DISC $(t+1)$ implies DISC (t) for $t \geq 2$. It is known that

$$e_t(V,V) \geq n(pn)^t \tag{2}$$

for any graph with average degree pn. This fact was first proved in [3] although it has been independently rediscovered by quite a few people [19, 21]. (Even the case of $t=3$ is an interesting exercise.) An elementary proof of (2) can be found in [2]. Here we will show that DISC(2) implies DISC(1).

Fact 11: DISC (2) $\Rightarrow$ DISC (1)**.**

Proof. For any subset X of the vertex set V of G, we have

$$\begin{aligned}e_2(X,X) &= \sum_{x,x' \in X} \sum_y e(x,y)e(y,x') \\ &= \sum_y (e(y,X))^2.\end{aligned}$$

From DISC (2) and Cauchy-Schwarz, we have

$$\begin{aligned}\frac{e(X,V)^2}{n} &\leq \sum_y e(y,X)^2 \\ &\leq p^2 n|X|^2 + o(p^2n^3).\end{aligned}$$

This implies that $|e(X,V)-pn|X||=o(pn^2)$ and

$$\sum_y (e(y,X)-p|X|)^2 \le \sum_y e(y,X)^2 - p^2n|X|^2 - 2p|X|(e(X,V)-pn|X|) = o(p^2n^3).$$

For any subset $Y \subset V$, we have

$$\begin{aligned} |e(X,Y)-p|X|\ |Y||^2 &\le n\sum_{y\in Y}(e(y,X)-p|X|)^2 \\ &\le n\sum_y (e(y,X)-p|X|)^2 \\ &= o(p^2n^4). \end{aligned}$$

which implies DISC (1). ∎

Example 4: CIRCUITS $(2t)$ $\not\Rightarrow$ CYCLE $(2t)$

Proof. There exists a graph on n vertices and $(1/2+o(1))n^{3/2}$ edges that contains no 4-cycles [15], but the number of 4-circuits for such a graph is $(1+o(1))n^2$. ∎

At present, we do not know the answer to the following:

Question: Is it true that " DISC $\Rightarrow$ EIG"?

6. Quasi-random classes

It is clear from the existence of examples like those in the preceeding section that in order to build a more extensive equivalence class of quasi-random properties, we will need to require additional hypotheses on the graph families. In this section, we will do this.

Consider the following *degree restriction* property DEG a graph family $\mathcal{G}_p$ might have:

DEG: For some absolute constant c, all vertices v of any $G(n)\in\mathcal{G}_p$ satisfy

$$\deg v < cpn.$$

Theorem 2. *If $\mathcal{G}_p$ satisfies DEG then for any positive integer t,*

$$\mathbf{DISC}(1) \Rightarrow \mathbf{DISC}(t)$$

Proof. This certainly holds for $t=1$. We proceed by induction. Fix $j \geq 1$ and assume the result holds for $t=j$. Thus, for any ϵ, $0<\epsilon<1$, we have for all X and Y,

$$|e_i(X,Y) - p^i n^{i-1}|X|\ |Y|| \leq \epsilon p^i n^{i+1} \tag{3}$$

for $1 \leq i \leq j$ and n sufficiently large. Consider the set

$$S_1 = \{z \in V\ :\ e_j(z,Y) \geq p^j n^{j-1}|Y| + \sqrt{\epsilon} p^j n^j\}.$$

From (3), we see that

$$|S_1| \leq \sqrt{\epsilon} n,\ e_j(S_1,Y) < (\epsilon + \sqrt{\epsilon}) p^j n^{j+1}.$$

Similarly, for

$$S_2 = \{z\ :\ e_j(z,Y) \leq p^j n^{j-1}|Y| - \sqrt{\epsilon} p^j n^j\}$$

we have

$$|S_2| \leq \sqrt{\epsilon} n,\ e_j(S_2,Y) \leq (\epsilon + \sqrt{\epsilon}) p^j n^{j+1}.$$

Thus,

$$\begin{aligned} e_{j+1}(X,Y) &= \sum_v e_1(X,v) e_j(v,Y) \\ &= \sum_{v \notin S_1 \cup S_2} e_1(X,v) e_j(v,Y) + \sum_{v \in S_1 \cup S_2} e_1(X,v) e_j(v,Y) \end{aligned}$$

For the first sum, we have

$$\begin{aligned} \sum_{v \notin S_1 \cup S_2} e_1(X,v) e_j(v,Y) &\leq \sum_{v \notin S_1 \cup S_2} e_1(X,v)(p^j n^{j-1}|Y| + \sqrt{\epsilon} p^j n^j) \\ &\qquad \text{by the definition of the } S_i \\ &\leq \sum_{v \in V} e_1(X,v)(p^j n^{j-1}|Y| + \sqrt{\epsilon} p^j n^j) \\ &\leq e_1(X,V)(p^j n^{j-1}|Y| + \sqrt{\epsilon} p^j n^j) \\ &\leq (pn|X| + \epsilon p n^2)(p^j n^{j-1}|Y| + \sqrt{\epsilon} p^j n^j) \\ &\qquad \text{for } n \text{ sufficiently large by hypothesis} \\ &\leq p^{j+1} n^j |X|\ |Y| + 3\sqrt{\epsilon} p^{j+1} n^{j+2}. \end{aligned}$$

For the second sum, we have

$$\begin{aligned}\sum_{v\in S_1\cup S_2} e_1(X,v)e_j(v,Y) &\le \sum_{v\in S_1\cup S_2} cpn\ e_j(v,Y) \\ &\qquad\qquad \text{by DEG} \\ &\le cpne_j(S_1\cup S_2,Y) \\ &\le cpn(p^j n^{j-1}|S_1\cup S_2|\ |Y| + \epsilon p^j n^{j+1}) \\ &\qquad\qquad \text{by the induction hyp.} \\ &\le cpn(p^j n^{j-1} 2\sqrt{\epsilon} n^2 + \epsilon p^j n^{j+1}) \\ &\qquad\qquad \text{by properties of the } S_i \\ &\le 3c\sqrt{\epsilon} p^{j+1} n^{j+2}.\end{aligned}$$

Combining these two estimates we obtain

$$e_{j+1}(X,Y) \le p^{j+1}n^j|X|\ |Y| + (3c+1)\sqrt{\epsilon}p^{j+1}n^{j+2}.$$

A similar argument shows that

$$e_{j+1}(X,Y) \ge p^{j+1}n^j|X|\ |Y| - c'\sqrt{\epsilon}p^{j+1}n^{j+2}$$

for a fixed constant c'. Thus,

$$|e_{j+1}(X,Y) - p^{j+1}n^j|X|\ |Y|| = o(p^{j+1}n^{j+2})$$

which implies that DISC$(j+1)$ holds, thus completing the induction step. This proves the theorem. ∎

Fact 12: DEG implies that for any $G(n)$ in $\mathcal{G}_p$,

$$\sum_{u,v} |P_t(u,v)| = (1+o(1)) \sum_{u,v} e_t(u,v),$$

where t is a fixed positive integer.

Proof. The number of t-walks in $G(n)$ is at least $(1+o(1))p^t n^{t+1}$ (see [3, 19, 21]). Let W denote a t-walk $v_0, v_1, \ldots, v_t$ in H which is not a t-path. Thus, $v_i = v_j$ for some $i \le j-2$, and W can be decomposed into three walks,

$v_i, v_{i-1}, \ldots, v_0$,

$v_i, v_{i+1}, \ldots, v_j$, and

$v_i, v_{j+1}, \ldots, v_t$, (since $v_i = v_j$).

For each choice of v_i, there are at most $(cpn)^i(cpn)^{j-1-i}(cpn)^{t-j} = (cpn)^{t-1} = c'(pn)^{t-1}$ choices for these walks. Hence, there are at most $c''_t p^{t-1} n^t$ choices for W, which is $o(p^t n^{t+1})$ since $pn \gg 1$. Hence the number P' of t-walks which are not t-paths is at most $o(p^t n^{t+1})$. ∎

We consider the following upper bound property U(t) on the walks of length $t \ge 2$ for a graph family $\mathcal{G}_p$.

U(t): $\mathcal{G}_p$ satisfies DEG, and for an absolute constant $c_0 > 1$, and for all vertices u and v of any $G(n) \in \mathcal{G}_p$, we have

$$e_{t-1}(u,v) < c_0 p^{t-1} n^{t-2}.$$

We remark that U(t) can only hold if $p > c_1 n^{-1+1/(t-1)}$ where c_1 depends on c_0. Obviously, any dense graph (with $p = c$) satisfies U(2). Because of DEG, it can be easily checked that if $\mathcal{G}_p$ satisfies U(t), then it must also satisfy U(t+1).

Theorem 3. *If $\mathcal{G}_p$ satisfies* U(t), $t \geq 2$, *then*

$$\textbf{CIRCUIT}(2t) \Leftrightarrow \textbf{CYCLE}(2t)$$

Proof. We consider $P' = M^*_{2t} - M_{2t}$ where M^*_{2t} is the number of $2t$-circuits and M_{2t} is the number of $2t$-cycles. It suffices to show that $P' = o(p^{2t} n^{2t})$.

Note that if H is a $2t$-circuit but not a $2t$-cycle, then we can choose two vertices u and v so that:
(1) H is the union of two t-walks joining u and v, and,
(2) at least one of the t-walks is not a t-path.
Thus,

$$\begin{aligned} P' &\leq 2t \sum_{u,v} e_t(u,v)(e_t(u,v) - |P_t(u,v)|) \\ &\leq 2tcp^t n^{t-1} \sum_{u,v} (e_t(u,v) - |P_t(u,v)|) && \text{since U(t) implies U(t+1)} \\ &\leq 2tcp^t n^{t-1} o(\sum_{u,v} e_t(u,v)) && \text{by Fact 12} \\ &\leq 2tcp^t n^{t-1} o(p^t n^{t+1}) \\ &= o(p^{2t} n^{2t}) \end{aligned}$$

∎

Theorem 4. *If $\mathcal{G}_p$ satisfies* U(t), $t \geq 2$, *then*

$$\textbf{DISC}(1) \Rightarrow \textbf{CIRCUIT}(2t)$$

Proof. We observe that the number of (ordered) $2t$-circuits is just

$$\sum_{u,v} e_t^2(u,v)$$

From Theorem 2, we know that DISC(t) holds so that for any $\epsilon > 0$, $e_t(V,V) = \sum_{u,v} e_t(u,v)$ satisfies

$$\left| \sum_{u,v} e_t(u,v) - p^t n^{t+1} \right| \leq \epsilon p^t n^{t+1} \tag{4}$$

for n sufficiently large. Hence,

$$\sum_{u,v}(e_t(u,v)-p^tn^{t-1})^2 = \sum_{u,v} e_t(u,v)^2 - 2p^tn^{t-1}\sum_{u,v} e_t(u,v) + (pn)^{2t}$$
$$= \sum_{u,v} e_t(u,v)^2 - (pn)^{2t} + \Delta$$

where $|\Delta| < 2\epsilon(pn)^{2t}$. Therefore, it will suffice to show that

$$\sum_{u,v}(e_t(u,v)-p^tn^{t-1})^2 = \epsilon''(pn)^{2t}$$

where $\epsilon'' \to 0$ as $\epsilon \to 0$.

For $u \in V$, define

$$Z_u = \{v \in V \ : |e_t(u,v) - p^tn^{t-1}| > \epsilon^{1/4}p^tn^{t-1}\}$$

and define

$$Z = \{u \in V \ : |Z_u| \geq \epsilon^{1/4}n\}.$$

Claim: $|Z| < \epsilon' n$ where $\epsilon' \to 0$ as $\epsilon \to 0$.

Assume for now the validity of the Claim (which we will prove shortly). Then

$$\sum_{u,v}(e_t(u,v)-p^tn^{t-1})^2$$
$$= \left(\sum_u \sum_{v\notin Z_u} + \sum_{u\notin Z}\sum_{v\in Z_u} + \sum_{u\in Z}\sum_{v\in Z_u}\right)(e_t(u,v)-p^tn^{t-1})^2$$

Now, we upper bound each of the three sums:

$$\sum_u \sum_{v\notin Z_u}(e_t(u,v)-p^tn^{t-1})^2 \leq n^2\epsilon^{1/2}(p^tn^{t-1})^2$$

by the definition of Z_u.

$$\sum_{u\notin Z}\sum_{v\in Z_u}(e_t(u,v)-p^tn^{t-1})^2 \leq |V\setminus Z|\epsilon^{1/4}n(c_0p^tn^{t-1})^2$$
$$\leq c_0^2\epsilon^{1/4}p^{2t}n^{2t}$$

since $|Z_u| < \epsilon^{1/4}n$ for $u \notin Z$ and $e_t(u,v) < c_0p^tn^{t-1}$ by U(t).

$$\sum_{u\in Z}\sum_{v\in Z_u}(e_t(u,v)-p^tn^{t-1})^2 \leq |Z|n(c_0p^tn^{t-1})^2$$
$$\leq c_0^2\epsilon^{1/4}p^{2t}n^{2t}$$

by the Claim and U(t). Since ϵ was arbitrary, the theorem will follow, provided that the claim is proved.

Proof of the Claim. From DISC(1) and DEG, we can assume that DISC$(t-1)$ holds using Theorem 2. Namely,

$$|e_{t-1}(X,Y) - p^{t-1}n^{t-2}|X|\;|Y|| \le \epsilon p^{t-1}n^t,$$

for n sufficiently large.

We first note that there is a set $Z \le 2\sqrt{\epsilon}n$ such that for all $x \notin Z$, we have

$$|e_{t-1}(x,V) - p^{t-1}n^{t-1}| \le \sqrt{\epsilon}p^{t-1}n^{t-1}. \tag{5}$$

The above fact can be proved in a very similar way as in the proof of Theorem 2 and we omit the proof.

For $u \notin Z$, we partition $W = \{v : e_{t-1}(u,v) > 0\}$ into sets W_i which are defined as follows:

$$W_i = \{v \;:\; |e_{t-1}(u,v) - w_i p^{t-1}n^{t-2}| \le \epsilon^{1/4} w_i p^{t-1}n^{t-2}\} \setminus \bigcup_{j<i} W_j$$

where $w_i = c_0(1+\epsilon^{1/4})^{-i}$ and $i = 0,1,\dots,c'$ where $c' = \lceil \log c_0 / \log(1+\epsilon^{1/4}) \rceil$. Here we use the assumption $e_{t-1}(u,v) < c_0 p^{t-1}n^{t-2}$ from U(t). From the definition of W_i, we have, for $u \notin Z$,

$$\left| e_t(u,V) - \sum_{i=0}^{c'} |W_i| w_i p^{t-1}n^{t-2} \right| \le \epsilon^{1/4} \sum_{i=0}^{c'} |W_i| w_i p^{t-1} n^{t-2} \tag{6}$$

Thus, from (5), we have

$$\begin{aligned} &\left| \sum_i |W_i| w_i p^{t-1}n^{t-2} - p^{t-1}n^{t-1} \right| \\ &\le |e_t(u,V) - p^{t-1}n^{t-1}| + \epsilon^{1/4} \sum_i |W_i| w_i p^{t-1}n^{t-2} \le 3\epsilon^{1/4}p^{t-1}n^{t-1} \end{aligned} \tag{7}$$

since $\sum w_i |W_i| \le n$ by the definition of W_i.

For each i, if $|W_i| \ge \epsilon^{1/4}n$, we define $U_i = \{v : e(W_i,v) > p|W_i| + \epsilon^{1/4}p|W_i|\}$ and $U_i' = \{v : e(W_i,v) < p|W_i| - \epsilon^{1/4}p|W_i|\}$. If $|W_i| < \epsilon^{1/4}n$, we define $U_i = U_i' = \emptyset$. Let W^* denote the union of all W_i with $|W_i| \le \epsilon^{1/4}n$. Clearly, we have $|W^*| \le c'\epsilon^{1/4}n$. Also, we define $U^* = \{v : e(W^*,v) > p|W^*| + \epsilon^{1/4}pn$ or $e(W^*,v) < p|W^*| - \epsilon^{1/4}pn\}$.

From DISC(1), we have $|U_i| < \epsilon^{1/2}n$, $|U_i'| < \epsilon^{1/2}n$ for all i, and $|U^*| < \epsilon^{1/2}n$. We take $U = \bigcup_i (U_i \cup U_i') \cup U^*$ so that

$$|U| \le \sum_{i=0}^{c'} (|U_i| + |U_i'|) + |U^*| \le 2(c'+1)\epsilon^{1/2}n \le \epsilon^{1/4}n$$

for $\epsilon < 1$ and n sufficiently large. It is enough to show that $Z_u \subset U$. That is, we want to show that $v \notin U$ satisfies

$$|e_t(u,v) - p^t n^{t-1}| < \epsilon^{1/4} p^t n^{t-1}.$$

We consider t-walks joining u and v, say, $u = v_0, v_1, \ldots, v_{t-1}, v_t = v$, with $v_{t-1} \in W_i$. The number of such walks is in the range of $(1 \pm \epsilon^{1/4}) p w_i |W_i| p^{t-1} n^{t-2}$ if $|W_i| > \epsilon^{1/4} n$.

The total number $e_t(u,v)$ of t-walks joining u and v satisfies

$$\begin{aligned} &\left| e_t(u,v) - \sum_i p |W_i| w_i p^{t-1} n^{t-2} \right| \\ &\leq \epsilon^{1/4} \sum_i p |W_i| w_i p^{t-1} n^{t-2} + (p|W^*| + \epsilon^{1/4} pn) c_0 p^{t-1} n^{t-2}. \end{aligned}$$

By (6) and (7), we have

$$\begin{aligned} |e_t(u,v) - p^t n^{t-1}| &\leq \left| e_t(u,v) - \sum_i p |W_i| w_i p^{t-1} n^{t-2} \right| \\ &\quad + p \left| \sum_i |W_i| w_i p^{t-1} n^{t-2} - p^{t-1} n^{t-1} \right| \\ &\leq \epsilon' p^t n^{t-1} \end{aligned}$$

This completes the proof of the claim. ∎

Although U(t) automatically holds for families of dense graphs, it might not be convenient to verify this property for specific families of sparse graphs. Here we consider another condition:

W(t): For all vertices v and any vertex subset S of $G(n) \in \mathcal{G}_p$, DEG holds and

$$\frac{1}{|S|} (e_t(v,S) - p^t n^{t-1} |S|)^2 = o(p^{2t} n^{2t-1} / \log n).$$

Theorem 5. *If $\mathcal{G}_p$ satisfies W(t), $t \geq 2$, then*

$$\textbf{DISC}(t) \Rightarrow \textbf{CIRCUIT}(2t).$$

Proof. Suppose that for $\epsilon > 0$,

$$\frac{1}{|S|} (e_t(u,S) - p^t n^{t-1} |S|)^2 < \epsilon p^{2t} n^{2t-1} / (2 \log n)$$

holds for all $u \in V$ and all $S \subset V$, and n sufficiently large. Fix $u \in V$ and consider $e_t(u,v)$ for each vertex v of G. Suppose we order the v's so that

$$\begin{aligned} e_t(u,v_1) \geq e_t(u,v_2) \geq \ldots \geq e_t(u,v_k) \geq p^t n^{t-1} > e_t(u,v_{k+1}) \\ \geq \ldots \geq e_t(u,v_n). \end{aligned}$$

By W(t), the sequence $e_t(u,v_i) - p^t n^{t-1}$, for $i = 1,\ldots,n$, is majorized by

$$m, \frac{m}{\sqrt{2}}, \ldots, \frac{m}{\sqrt{i}}, \ldots, \frac{m}{\sqrt{n}}$$

where

$$m = \sqrt{\frac{\epsilon p^{2t} n^{2t-1}}{2\log n}}.$$

Thus, we have

$$\begin{aligned} \sum_v (e_t(u,v) - p^t n^{t-1})^2 &\leq 2m^2 \sum_{i=1}^{n} \frac{1}{i} \\ &< 2m^2(1+\log n) \\ &\leq 2\epsilon p^{2t} n^{2t-1} \end{aligned}$$

for n sufficiently large. Hence, the total number of $2t$-circuits $M^*(2t)$ is upper-bounded by

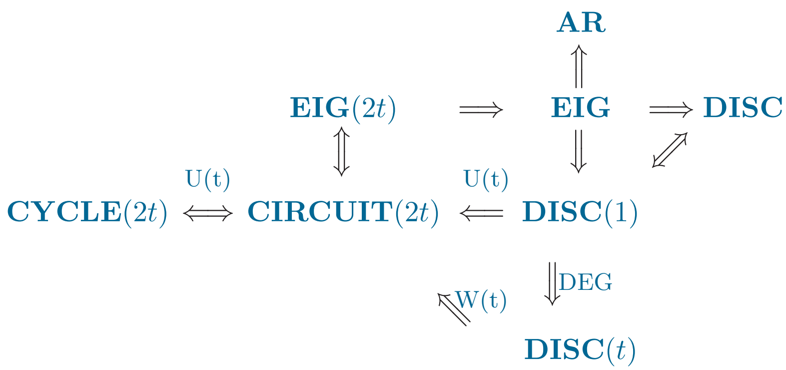
$$\begin{aligned} M^*(2t) &= \sum_{u,v} e_t^2(u,v) \\ &= p^{2t}n^{2t} + \sum_{u,v}(e_t(u,v) - p^t n^{t-1})^2 + 2\sum_{u,v}(e_t(u,v) - p^t n^{t-1})p^t n^{t-1} \\ &\leq (1+\epsilon')p^{2t}n^{2t} \end{aligned}$$

where we can take $\epsilon' = \epsilon + \sqrt{8\epsilon}$. On the other hand, we have

$$\begin{aligned} M^*(2t) &= \sum_{u,v} e_t(u,v)^2 \\ &\geq \frac{1}{n^2}\left(\sum_{u,v} e_t(u,v)\right)^2 \\ &\geq (1+o(1))p^{2t}n^{2t} \end{aligned}$$

by using DISC($2t$). Thus, since $\epsilon > 0$ was arbitrary, the proof is complete. ∎

The following figure summarizes the results of Theorems 1–5, for $t \geq 2$.



As an immediate consequence of the above diagram, we have the following:

Theorem 6. *Suppose for some constant* $c > 0$, $p > cn^{-1+1/(t-1)}$ *where* $t \geq 2$. *For any family* $\mathcal{G}_p$ *of graphs satisfying either* U(t) *or* W(t) *the following properties are all equivalent:*
DISC, DISC(1), DISC(t), CIRCUIT($2t$), EIG and EIG($2t$).

Let us call a class $\mathcal{G}_p$ which satisfies either U(t) or W(t) *t-quasi-random* if, in addition, it also satisfies any one of the properties DISC, DISC(1), DISC(t), CIRCUIT($2t$), EIG or EIG($2t$) (which we will also refer to as t-quasi-random properties). Of course, this means that $\mathcal{G}_p$ must in fact satisfy *all* of these properties. The quasi-random families for dense graphs with $p = c$, for some constant c (as described in Section 2), are just 2-quasi-random. We call a family $\mathcal{G}_p$ *strongly t-quasi-random* if in fact it satisfies U(t) in addition to being t-quasi-random.

7. Properties of t-quasi-random graphs

We now mention several properties of t-quasi-random families with $t \geq 2$.

Theorem 7. *Suppose a family* $\mathcal{F} \subseteq \mathcal{G}_p$ *of graphs is t-quasi-random. Then* $\mathcal{F}$ *is* $(t+1)$*-quasi-random.*

Proof. First we observe that U(t) $\Rightarrow U(t{+}1)$ and W(t) $\Rightarrow W(t{+}1)$. However, EIG($2t$) $\Rightarrow$ EIG($2t+2$) now follows from Fact 9. ∎

If H is a graph, we denote by $H^{(t)}$ the *t-subdivision* of H, the graph formed from H by replacing each of H by a path of length t, (i.e., subdividing each edge of H with $t-1$ additional vertices).

Theorem 8. *If $\mathcal{F} \subset \mathcal{G}_p$ is strongly t-quasi-random, then for any fixed graph H and $G(n) \in \mathcal{F}$,*

$$\#\{H^{(t)} \subset G(n)\} = (1+o(1))n^{v(H^{(t)})}p^{e(H^{(t)})}.$$

Proof. First we note that CIRCUIT$(2t)$ and DISC(t) imply that

$$\sum_{x,y}(e_t(x,y) - p^t n^{t-1})^2 = o((pn)^{2t}).$$

This in turn implies the following property: For all but $o(n^2)$ pairs of vertices x and y in $G(n)$, we have

$$e_t(x,y) = (1+o(1))p^t n^{t-1}.$$

For a fixed $\epsilon > 0$, let E' denote the set of edges of $G(n)$ satisfying:

$$E' = \{\{u,v\} \ : \ |e_t(u,v) - p^t n^{t-1}| > \epsilon p^t n^{t-1}.\}$$

Now, for a given graph H, there are $n^{v(H)}$ maps π from the vertices of H to the vertices of $G(n)$. For any edge $\{x,y\}$ of H, there are $e_t(\pi(x),\pi(y))$ ways to map the subdivision path joining x and y in $H^{(t)}$. We partition the set of π's into two parts, S and T, where S contains those π's that map some edge of H to an edge in E', and T is the complement of S. Thus,

$$\begin{aligned}\#\{H^{(t)} \subset G(n)\} &= \sum_{\pi} \prod_{\{x,y\}\in E(H)} e_t(\pi(x),\pi(y)) \\ &= \sum_{\pi\in S} \prod_{\{x,y\}\in E(H)} e_t(\pi(x),\pi(y)) + \sum_{\pi\in T} \prod_{\{x,y\}\in E(H)} e_t(\pi(x),\pi(y)).\end{aligned}$$

First, we derive an upper bound for the following sum:

$$\sum_{\pi\in S} \prod_{\{x,y\}\in E(H)} e_t(\pi(x),\pi(y)).$$

From U(t), we know that

$$e_t(\pi(x),\pi(y)) < cp^t n^{t-1}.$$

For each edge q in E', there are at most $2n^{v(H)-2}$ maps π mapping an edge of H to q. Thus S contains at most $2|E'|n^{v(H)-2} = o(n^{v(H)})$ elements by the definition of E'. Hence,

$$\begin{aligned}&\sum_{\pi\in S} \prod_{\{x,y\}\in E(H)} e_t(\pi(x),\pi(y)) \\ &\le \sum_{\pi\in S}(cp^t n^{t-1})^{e(H)} \\ &= o(n^{v(H)})(cp^t n^{t-1})^{e(H)} \\ &= o(n^{v(H^{(t)})}p^{e(H^{(t)})}).\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\sum_{\pi\in T}\prod_{\{x,y\}\in E(H)} e_t(\pi(x),\pi(y))\\
&=\sum_{\pi\in T}(1+o(1))(p^t n^{t-1})^{e(H)}\\
&=\sum_{\pi}(1+o(1))(p^t n^{t-1})^{e(H)}\\
&=(1+o(1))n^{v(H)}(p^t n^{t-1})^{e(H)}\\
&=(1+o(1))n^{v(H^{(t)})}p^{e(H^{(t)})}
\end{aligned}$$

for n sufficiently large. ∎

A natural question is to relate t-quasi-random properties to the following property concerning subgraph containment: For a t-quasi-random family $\mathcal{F}\subset\mathcal{G}_p$, is it true that for $G(n)\in\mathcal{F}$ and any fixed graph H,

$$\#\{H\subset G(n)\}=(1+o(1))n^{v(H)}p^{e(H)}?$$

Indeed, this is true for many families of H, e.g., $H=C_{2t}$ (by CYCLE($2t$)), and t-subdivisions of a given graph (by the above theorem) as well as any H when $t=2$. However, the answer to the above question can be negative for certain graphs. For example, suppose we take H' to consist of a triangle joined to a path on $v(H')$–2 vertices. The number of containments of such H' in a t-quasi-random graph can be quite different from the value above, e.g., the Ramanujan graphs as described in the next section can have relatively large girth and so cannot contain such H' as a subgraph.

Problem: For a t-quasi-random family $\mathcal{F}\subset\mathcal{G}_p$, is it true that for any $G(n)\in\mathcal{F}$ and for any fixed graph H with girth at least $2t$,

$$\#\{H\subset G\}=(1+o(1))n^{v(H)}p^{e(H)}?$$

8. Examples of t-quasi-random graph families

There are a variety of constructions for sparse quasi-random graph families which have appeared in the literature. We mention several of these here.

To begin, Thomason [26–28] describes a number of constructions, which, in particular, turn out to be t-quasi-random for some t. One of these are the so-called Erdős–Rényi graphs, which are the following. Let q be a prime power and let $V(G)$ be the points of a projective plane over $GF(q)$. Thus, $n=|V(G)|=q^2+q+1$. Join $x=(x_0,x_1,x_2)$ to $y=(y_0,y_1,y_2)$ if $x_0y_0+x_1y_1+$

$x_2y_2 = 0$ in $GF(q)$. Then it is true that DISC holds for this family with $p = (q+1)/(q^2+q+1)$. In fact, a much stronger inequality holds, namely, for any $X \subset V(G)$,

$$\left| e(X) - p\binom{|X|}{2} \right| \leq 2(pn)^{1/2}|X|.$$

It is known that the graphs in this family do not contain C_4's. It is not hard to check that this family satisfies W(3). Hence, the Erdős–Rényi graphs are 3-quasi-random but not 2-quasi-random.

For this class of graphs, we have $e(G) \sim cn^{3/2}$. This is also the bound for many of the constructions given by Thomason [27] for his $(p,\alpha)-$jumbled graphs. To push $e(G)$ down below $n^{3/2}$ we need more powerful constructions, several of which we now list.

Coset graphs $C_{P,t}$ (introduced in Chung [4]) are defined as follows. For a prime P and an integer $t \geq 2$, let $GF(P^t)^*$ denote the multiplicative group of the finite field $GF(P^t)$. For a fixed generator g of $GF(P^t)^*$, let $\lambda_g(z) \in \{1, 2, \ldots, P^t - 1\}$ denote the logarithm (to the base g) of $z \in GF(P^t)$, i.e., $g^{\lambda_g(z)} = z$. Define the vertex set $V(C_{P,t})$ to be $\{1, 2, \ldots, P^t - 1\}$. For $a, b \in V(C_{P,t})$, $\{a, b\}$ is an edge of $C_{P,t}$ provided $a + b \in G$ where $G = \{\lambda(y) : y \in$ coset $g + GF(P)\}$. It follows by an eigenvalue estimate of Katz[18] that $|\lambda_2| \leq (t-1)\sqrt{P}$ for the graph $C_{P,t}$. Since $C_{P,t}$ is regular of degree P, then the family $\{C_{P,t}\}$ satisfies EIG(2t) for t fixed as $P \to \infty$.

To see that $\{C_{P,t}\}$ satisfies W(t), we consider $e_t(u, S)$ for a subset S. Let f_S denote the characteristic function of S as defined in Section 4, Fact 3. Suppose we write $f_S = \sum_i a_i \bar{e}_i$ where $\bar{e}_i$ are eigenvectors of the adjacency matrix forming an orthonormal basis, with $\bar{e}_1$ the eigenvector with the largest eigenvalue. Also, we let f_u denote the characteristic function of $\{u\}$ and write $f_u = \sum_i b_i \bar{e}_i$. We have

$$\begin{aligned} |\langle f_u, A^t f_S \rangle - \lambda_1^t a_1 b_1| &\leq |\lambda_2|^t \sum_i |a_i b_i| \\ &\leq |\lambda_2|^t \left(\sum_i a_i^2 \right)^{1/2} \left(\sum_i b_i^2 \right)^{1/2} \\ &\leq c(pn)^{t/2} \sqrt{|S|}. \end{aligned}$$

We here use the fact that $C_{P,t}$ is regular with eigenvector $\bar{e}_1 = \bar{u}$, the vector with all coordinates equal to $1/\sqrt{n}$. Since $e_t(u,S) = \langle f_u, A^t f_S \rangle$, $a_1 \sim |S|/\sqrt{n}$ and $b_1 \sim 1/\sqrt{n}$, we have

$$\begin{aligned} \frac{1}{|S|}(e_t(u,S) - p^t n^{t-1}|S|)^2 &\leq (1 + o(1))c^2(pn)^t \\ &= o(p^{2t}n^{2t-1}/\log n) \end{aligned}$$

for n sufficiently large. Thus, W(t) is satisfied and the family $\{C_{P,t}\}$ is t-quasi-random.

We remark that these graphs have loops and multiple edges but the various quasi-random properties still make sense. It is not difficult to show that if we modify these graphs by removing loops and multiple edges then the resulting graphs still satisfy the various conditions needed for t-quasi-randomness.

Finally, we mention the celebrated Ramanujan graphs constructed by Lubotzky, Phillips and Sarnak [20]. We omit the detailed descriptions of these striking graphs here but rather just list some of their parameters. Specifically, there is a Ramanujan graph $X^{P,Q}$ when P and Q are primes congruent to 1 modulo 4 and the Legendre symbol $(P/Q) = 1$. Then $v(X^{P,Q}) = Q(Q^2-1)/2$ and $X^{P,Q}$ is $(P+1)$-regular, which implies $e(X^{P,Q}) = (P+1)Q(Q^2-1)/4$. Also, we have $\lambda_2 \le 2\sqrt{P}$ so that $X^{P,Q}$ satisfies EIG provided $Q \to \infty$ as $P \to \infty$. Now, suppose we choose t such that $Q^3 \log Q = o(P^t)$. We can use exactly the same proof as above (for the coset graphs) to show that W(t) is satisfied. This shows that Ramanujan graphs $X^{P,Q}$ are t-quasi-random graphs families if $Q^3 \log Q = o(P^t)$.

9. Concluding remarks

We mention here a number of problems which we feel merit further attention.

To begin, is there a t-quasi-random condition which depends on $\#\{H < G\}$, which denotes the number of *induced* subgraphs of $G \in \mathcal{G}_p$ which are isomorphic to some fixed graph H, when p is $o(1)$? For constant p, this is the case. In fact, it is conjectured in this case that if $e(G(n)) \ge (1+o(1))pn^2/2$ and $\#\{H_r < G_n\} \le (1+o(1))p^{e(H)}(1-p)^{\binom{r}{2}-e(H)}n^r$ where H is *any* fixed bipartite graph on r vertices having at least one cycle, then this already implies that G is 2-quasi-random.

Thomason [27] shows that when the graphs in question have a stronger "jumbled" condition (in particular, the error term in the discrepancy is of lower order than the main term), then if $e(G_n) > n^{2-\epsilon_r}$ for a sufficiently small $\epsilon_r > 0$, then $\#\{H_r < G_n\}$ is asymptotically what it should be.

It would be desirable to have a quantitative version of t-quasi-randomness, i.e., one which applies to individual graphs as opposed to families of graphs. For $p = 1/2$, for example, one can define *dev* G_n, the *deviation* of a graph $G(n)$, by

$$\text{dev } G_n = \frac{1}{n^4} Tr(J - 2A(G_n))^4$$

where $A(G_n)$ is the adjacency matrix of G_n, and J is the $n \times n$ matrix of all 1's. It is known [12] that the family G_n is 1/2-quasi-random if and only if dev $G_n = o(1)$ as $n \to \infty$. In fact, as dev G_n get closer to 0, each of the quasi-random parameters gets closer to its expected value (for a random graph with $p = 1/2$). In a subsequent paper we will define and study the appropriate analogue for sparse graphs.

We remark that for fixed p (and in particular, $p=1/2$) there is a powerful extension of most of these ideas to k-uniform hypergraphs [5, 10]. What are the corresponding extensions in the sparse case? In particular, is there a hypergraph version of the recent result of Prömel and Rödl [22] which shows that "non-Ramsey" graphs are universal?

Finally, as mentioned earlier, does DISC imply EIG for every family in $\mathcal{G}_p$?

These are just a few of the many intriguing questions which remain. We hope to address some of them in the near future.

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